Building Algebraic Models of Logics within Mizar Mathematical Library
(Extended Abstract)

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Abstract. Among many areas of the research conducted by Helena Rasiowa within mathematical logic, topology, and mathematical foundations of computer science we pay special attention to the development of the algebraic approach to logics. The monograph "The Mathematics of the Metamathematics" by Helena Rasiowa and Roman Sikorski was a great source of inspiration for Andrzej Trybulec in his development of the Mizar system – the environment for formalizing mathematics using computer for the verification of proofs. In our paper we give some further examples showing the influence of Rasiowa's papers devoted to algebraic models of logic and the connections between logic and topology on the Mizar Mathematical Library – the repository of mathematical knowledge verified by means of computer proof-checker. As a main illustrative case we use Nelson algebras (or, after Rasiowa and Białynicki-Birula, quasi-pseudo-Boolean algebras).

Keywords: formalization of mathematics, Mizar Mathematical Library, Nelson algebra

Recent achievements of Thomas Hales (Kepler conjecture), Georges Gonthier (four color theorem and Feit-Thompson proof of the odd-order theorem) and William McCune (Robbins conjecture) show that the usage of computer provers and proof checkers helps mathematicians to prove hard mathematical problems unsolved for many years, even many centuries. The key problem is to make automatically generated proofs understandable for a wider audience of mathematicians. In order to achieve this goal we should think about a common formal language which has enough expressive power being close to a natural language at the same time. The Mizar system offers a framework for formal verification, analysis and refactoring proofs. The proofs are a part of the Mizar Mathematical Library – a collection of mathematical texts written with the use of especially designed formalism based on Tarski-Grothendieck set theory and first-order predicate logic. This computer repository is a common platform for researchers within various areas of mathematics.

In the same time, via parallel accessibility to distinct repositories of mathematical knowledge, new light can be shed on classical results. Taking rough set theory as an example, it is yet classical result about the connection between rough approximation operators and topological closure and interior operators, but some additional linking (with Isomichi classification of subsets inspired partly by Kuratowski closure-complement problem) was discovered automatically by means of tools available in the Mizar Mathematical Library.

The study of the interconnections between algebra and logic, carried out by Helena Rasiowa, as a kind of consequence of the foundational crisis of mathematics raised at the beginning of the twentieth century, is clearly visible in the monograph *"The Mathematics of the Metamathematics"*, a joint work with R. Sikorski [15]. This book was a great source of inspiration for the developer of the Mizar system, Andrzej Trybulec[1].

Among underlined passages are the following:

> *"As we have mentioned, all known mathematical theorems can be derived from the set of axioms of formalized set theory. We shall not prove this in detail. [...] Since, in formalized theories, the process of derivation of formulas from axioms consists of mechanical operations on formulas, [...] the chief result of this section is rather shocking: all known mathematical theorems can be derived from the set of axioms of formalized set theory and from the set of all logical axioms [...] by means of purely mechanical operations called the rules of inference [...]."*

which is not very surprising now; remember it was written in 1962, some thirty years after showing that Hilbert's program (of reducing the consistency of all the mathematics in terms of simpler systems) is unattainable for key areas of mathematics. Also the term *mechanical operations* is even less cryptic now.

The Mizar system was developed as a tool for ordinary mathematicians. The corresponding language uses classical first-order logic with schemes with all its expressive power. The axioms of Tarski-Grothendieck, close to Zermelo-Fraenkel with built-in Axiom of Choice serves as underlying set theory – the base of the Mizar Mathematical Library.

> *"On the other hand, it should be emphasized that this mechanical method of deducing some mathematical theorems has no practical value because it is too complicated in practice."* (p. 201)

These words were written five years before the first description of Automath by de Bruijn as a formal language for automatizing mathematics. Obviously, first attempts to use this paradigm were hard and complicated, although quite successful (checking fragments of Landau's *Grundlagen der Analysis*).

The earliest developments of classical propositional calculus were formalized in Mizar by Trybulec and Rudnicki [16]. Even if the work gives a direct construction of a free algebra of the language, the pure set-theoretical setting can

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[1] The author owns the copy of the book by Rasiowa and Sikorski annotated by A. Trybulec, and some direct consequences of thoughts contained in Rasiowa's monograph reflected in the implementation of the system were mentioned many times by him in private communication.

be named today an old-fashioned one. First of all, it does not really offer the possibility of multiple generalizations via extensions of the language (connectives are tightly bound with consecutive natural numbers, hence any revision of the approach needs at least a kind of renumeration of objects). Also the use of lattices or any similar algebra used to describe the language was completely absent. This was a part of more general trend: the algebraization of logic was not clearly visible in MML from its beginnings, although Boolean algebras were a basic algebraic model used in the Mizar Mathematical Library (apart from abstract algebras which were used obviously for quite distinct purpose), and also general lattices (which also appear in Rasiowa and Sikorski's monograph very early). Consequently, Hilbert positive propositional calculus was implemented by us in [5] in the new, algebraic, spirit – building some bridges, however, with the abovementioned approach to first order language.

Nelson algebras were first studied by Rasiowa and Białynicki-Birula [2] under the name N-lattices or quasi-pseudo-Boolean algebras. Later, in investigations by Monteiro and Brignole [4] and [3] the name "Nelson algebras" was adopted – which is now commonly used to show the correspondence with Nelson's paper [11] on constructive logic with strong negation.

By a Nelson algebra we mean an abstract algebra

$$\langle L, \top, -, \neg, \rightarrow, \Rightarrow, \sqcup, \sqcap \rangle$$

where L is the carrier, $-$ is a quasi-complementation (Rasiowa used the sign $\sim$, but in Mizar "$-$" should be used to follow the approach described in [6]), $\neg$ is a weak pseudo-complementation, $\rightarrow$ is weak relative pseudo-complementation and $\Rightarrow$ is implicative operation. $\sqcup$ and $\sqcap$ are ordinary lattice binary operations of supremum and infimum.

We formally developed the definition and basic properties of these algebras according to [13] and [14]. In a natural way, we started with preliminaries on quasi-Boolean algebras (i.e. de Morgan bounded lattices). Later we gave the axioms in the form of Mizar adjectives with names corresponding with those in [14]. As our main result we give two axiomatizations (non-equational and equational) and the full formal proof of their equivalence.

We provide the Mizar scripts offering full formal translation (with proofs) of all items from Th. 1.2 and 1.3 (and the itemization is given in the text) and also the proof of Th. 2.1 p. 75 [13]. There are slight differences in the enumeration of formulas between [13] and [14] and we wanted to find a reasonable compromise allowing later to give a correspondence with virtually all parts from the latter book, including representation theorem for Nelson algebras.

To give the impression of how the development looks like, we quote some Mizar excerpts. In order to use as much as possible structures existing in MML, we start with the signature of algebras under consideration:

```
definition
  struct (OrthoLattStr) NelsonStr
               (# carrier -> set,
                    unity -> Element of the carrier,
```

```
          Compl, Nnegation -> UnOp of the carrier,
       Iimpl, impl, L_join, L_meet -> BinOp of the carrier #);
end;
```

This means that we deal with two negations (one of these is `Compl` inherited from the theory of ortholattices, used, e.g., in the solution of the Robbins problem), two implications, and of course lattice supremum and infimum together with the lattice upper bound (`unity`).

Then, the list of Mizar attributes describing all axioms follows, with the example of (N11):

```
definition let L be non empty NelsonStr;
  attr L is satisfying_N11 means
:: NELSON_1:def 18
    for a being Element of L holds
      -!a < a;
end;
```

We can conclude with a little bit long definition of Nelson algebra:

```
definition
  mode Nelson_Algebra is satisfying_A1 satisfying_A1b satisfying_A2
       satisfying_N3 satisfying_N4 satisfying_N5 satisfying_N6
       satisfying_N7 satisfying_N8 satisfying_N9 satisfying_N10
       satisfying_N11 satisfying_N12 satisfying_N13 bounded Lattice-like
         non empty NelsonStr;
end;
```

where numbers correspond to the numbering scheme proposed by Rasiowa [13]. To be honest, more self-explanatory names still could be proposed.

Thanks to the flexibility of used approach, this algebra can be freely used in the theory of rough sets (with the connection with de Morgan algebras). Of course, basic examples of such defined algebras were constructed, and quite natural properties could be proven (most of them automatically, by means of automated theorem provers).

```
::NdR RasiowaNonClassical: p 70 1.3 (25)
theorem :: NELSON_1:46  :: (2.26)
  ((a => (b => c)) => ((a => b) => (a => c))) = Top L;
```

Apart from the strict Mizar source, we will present not only two equivalent axiomatizations of Nelson algebras, which is already available in the repository of Mizar texts, but also sketch the way of injecting theory of rough sets (also present widely in MML) into the algebraic apparatus, offering some automatic unification mechanisms between objects from distinct, although isomorphic in some sense, worlds. Until now, such connection was not formalized in full extent, although papers on this connection are well known [12].

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