Remarks on membership in granular computing within rough mereological framework

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Abstract

We address the problem of the criterion for membership in granules defined within rough mereology.

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1 Introduction: mereology, rough mereology

Our universa of things are assumed to be finite. For a universe U, we single out the sub-universe of individual things $I(U)$ by the Leśniewski ontological criterion [4] cf. [11], [2]:

$$x\ is\ y \Leftrightarrow \exists z.(z\ is\ x) \wedge \forall\ z.(z\ is\ x \Rightarrow\ z\ is\ y) \wedge \forall\ z, w.(z\ is\ x \wedge w\ is\ x \Rightarrow z\ is\ w),$$

which states what is understood by the statement $xisy$: the thing x belongs in the collection y; in particular, $x\ is\ x$ means that x is an individual thing.

On the sub-universe $I(U)$, the primitive notion of mereology due to Leśniewski [3], [12] of a *part* is defined. Part is a binary relation, denoted with $\pi(x, y)$, read as *x is a part of y* which is required to be:

M1 *Irreflexive*: For each $x \in I(U)$, it is not true that $\pi(x, x)$.

M2 *Transitive*: For each triple x, y, z of things in $I(U)$, if $\pi(x, y)$ and $\pi(y, z)$, then $\pi(x, z)$.

By M1, part means a proper part, distinct from the whole. The notion of an ingredient, $ingr(x, y)$ in symbols, accounts for the whole as well:

$$ingr(x, y) \Leftrightarrow \pi(x, y) \vee x = y. \tag{1}$$

Two things x, y *overlap*, symbolically $Ov(x, y)$ when there is a thing common to both as an ingredient:

$$Ov(x, y) \Leftrightarrow \exists z.ingr(z, x) \wedge ingr(z, y). \quad (2)$$

The notion of overlap is instrumental in the next axiom of mereology, which renders a deduction mechanism for mereology:

M3 For things x, y:
$ingr(x, y) \Leftrightarrow \ \forall t.[ingr(t, x) \Rightarrow \exists z.ingr(z, y) \wedge Ov(t, z)]$.

The aggregation mechanism is provided for mereology by the class forming operation [3] which allows for merging individual things from any non-vacuous concept into the individual thing. For a non-vacuous concept V, the *class of V*, symbolically denoted $Cls(V)$ is defined by the following conditions:

C1 $x \ in \ V \Rightarrow \ ingr(x, Cls(V))$.

C2 $ingr(x, Cls(V) \Rightarrow \exists z.[z \ in \ V \wedge \ Ov(x, z)]$.

Those notions are in analogy to the set–theoretic notions: when part is replaced with proper subset, then ingredient becomes subset and the class becomes the union of sets.

The exact composition scheme set by part on individual things needs a softening for purposes of rendering the approximate relations of containment. To this purpose rough mereology answers [10], [8] cf. [13]. The primitive notion of rough mereology is *rough inclusion*, $\mu(x, y, r)$ in symbols, read as *x is a part of y to a degree of r*.

Requirements for a rough inclusion are:

RM1 $\mu(x, x, 1)$.

RM2 $\mu(x, y, 1) \Leftrightarrow ingr(x, y)$.

RM3 $\mu(x, y, 1) \wedge \mu(z, x, r) \Rightarrow \ \mu(z, y, r)$.

RM4 $\mu(x, y, r) \wedge s \leq r \Rightarrow \ \mu(x, y, s)$.

2 Rough inclusions

We point here to some methods for inducing rough inclusions on universes of individual things. We begin with *residual rough inclusions* induced by continuous t-norms. We recall that a t-norm (triangular norm) cf. [6],[1], [5] is a function $T : [0, 1]^2 \rightarrow [0, 1]$ which does satisfy the conditions:

T1 $T(x, y) = T(y, x)$.

T2 $T(x, T(y, z)) = T(T(x, y), z)$.

T3 $x_1 > x_2 \Rightarrow T(x_1, y) \geq T(x_2, y)$.

T4 $T(x, 1) = x, T(x, 0) = 0$.

In addition, T may be continuous and may satisfy the condition

T5 $T(x, x) < x$ for $x \neq 0, 1$.

For a continuous t-norm T, we recall the definition of the *T-residuum*, $\Rightarrow_T$ in symbolic notation:

$$x \Rightarrow_T y = argmax_r[T(x, r) \leq y]. \tag{3}$$

We define a T-residual rough inclusion μ_T:

$$\mu_T(x, y, r) \Leftrightarrow x \Rightarrow_T y \geq r. \tag{4}$$

Proposition 1 *Equation 4 defines a rough inclusion.*

Proof 1 *For RM1, $\mu_T(x, y, 1)$ means that $x \Rightarrow_T y = 1$, hence, $x \leq y$, i.e., $ingr_M(x, y)$. For RM2, assume that $\mu_T(x, y, 1)$ and $\mu_T(z, x, r)$, hence (i) $x \leq y$ (ii) $z \Rightarrow_T x \geq r$, i.e., $T(z, r) \leq x$. By T3, $T(z, r) \leq y$, hence, by RM1, $z \Rightarrow_T y \geq r$. RM3 is manifestly satisfied.*

We resort to the classical t-norms of minimum, product and Łukasiewicz which give rise to *fuzzy logics*:

$$M(x, y) = min\{x, y\}, \tag{5}$$

$$P(x, y) = x \bullet y, \tag{6}$$

$$L(x, y) = max\{0, x + y - 1\}. \tag{7}$$

It is manifest that for each T-residual rough inclusion μ_T: $x \leq y \Rightarrow \mu_T(x, y) = 1$, hence, it suffices to consider the case $x > y$ only and in this case t-norms M, P, L do induce the following residual implications:

$$x \Rightarrow_M y = y, \tag{8}$$

$$x \Rightarrow_P y = \frac{y}{x}, \tag{9}$$

$$x \Rightarrow_L y = min\{1, y - x + 1\}, \tag{10}$$

with the corresponding rough inclusions.

We recall the important transitivity property of T-residual rough inclusions.

Proposition 2 *For each continuous t–norm T, the rough inclusion μ_T is T-transitive: if $\mu_T(x, y, r), \mu_T(y, z, s)$, then $\mu_T(x, z, T(r, s))$.*

Proof 2 *$\mu_T(x, y, r)$ is equivalent to $T(x, r) \leq y$, and, $\mu_T(y, z, s)$ is equivalent to $T(y, s) \leq z$. By coordinate–wise monotonicity of T, T3, it follows that $T(T(x, r), s) \leq z$, and, by associativity of T, T2, one obtains $T(x, T(r, s)) \leq z$, hence, $\mu_T(x, z) \geq T(r, s)$*

Following Proposition 2, we call a rough inclusion μ $f - transitive$ when there exists a function $f : [0, 1]^2 \to [0, 1]$ which is symmetrical,increasing in each coordinate, and, satisfying the condition $f(1, x) = x$ and the following property holds

$$\mu(x, y, r) \wedge \mu(y, z, s) \Rightarrow \mu(x, z, f(r, s)). \tag{11}$$

Rough residual inclusions are defined so far on the unit interval $[0, 1]$. Given a finite universe of individual things $I(U)$, let $\phi : U \to [0, 1]$ be an injection and let the rough inclusion μ_ϕ be defined as follows:

$$\mu_\phi(x, y, r) \Leftrightarrow \ \mu(\phi(x), \phi(y), r) \tag{12}$$

for each rough inclusion μ on the interval $[0, 1]$.

We recall that an information system in the sense of Pawlak [7] is a triple $IS = (U, A, V)$ where U is a finite universe of things (objects), A is a finite set of *attributes* and V is a set of *values*. Each attribute a in the set A is a mapping from the universe U into the value set V. We define in this setting two sets: $DIS(x, y), IND(x, y)$ as follows:

$$DIS(x, y) = \{a \in A : a(x) \neq a(y)\}, \tag{13}$$

$$IND(x, y) = \{a \in A : a(x) = a(y)\}. \tag{14}$$

We recall that things x, y with $DIS(x, y) = \emptyset$ are regarded as identical.

We aim at rough inclusions on the universe U, and, to this end, we recall our construction [8].

We recall that the t-norms P, L are up to the isomorphism only t-norms which satisfy the condition T5 (they are called *archimedean*) and as such they admit a Hilbert-style representation cf. [5]:

$$T(x, y) = g_T(f_T(x) + f_T(y)), \tag{15}$$

where f_T is an inverse automorphism on the interval [0,1], i.e., a decreasing bijection with $f(1) = 0$, whereas g_T is the *pseudoinverse* to f_T cf. [5]. In particular, for the Łukasiewicz t-norm L, one has $f_L(x) = 1 - x, g_L(y) = 1 - y$ for $x, y \in [0, 1]$.

For an archimedean rough inclusion T, we define the rough inclusion μ_T^I by means of:

$$\mu_T^I(x, y, r) \Leftrightarrow g_T(\frac{|DIS(x, y)|}{|A|}) \geq r. \tag{16}$$

Then, it is true that

Proposition 3 *μ_T^I is a rough inclusion with the associated ingredient relation of identity and the part relation empty.*

Proof 3 *We consider t-norm L, with $g_L(y) = 1 - y$, i.e, $g^{-1}(1) = 0$ so $\mu_T^I(x, y, 1)$ implies $DIS(x, y) = \emptyset$, which, by our assumption, is the identity $=$. This verifies the condition RM1, the rest follow along standard lines.*

The rough inclusion μ_L^I is given by means of the formula:

$$\mu_L^I(x, y, r) \Leftrightarrow 1 - \frac{|DIS(x, y)|}{|A|} \geq r. \tag{17}$$

As $IND(u, v) = A \setminus DIS(u, v)$, we obtain a new form of (17)

$$\mu_L^I(x, y, r) \Leftrightarrow \frac{|IND(u, v)|}{|A|} \geq r. \tag{18}$$

The formula (18) witnesses that the reasoning based on the rough inclusion μ_L^I is the probabilistic one. At the same time, we have given a logical proof for formulas like (18) that are very frequently applied in Data Mining and Knowledge Discovery, also in rough set methods in those areas, see. It also witnesses that μ_L^I is a generalization of indiscernibility relation to the relation of partial indiscernibility.

In case of the product t–norm P, the formula (16) specifies to

$$\mu_P^I(u, v, r) \Leftrightarrow exp(-\frac{|DIS(u, v)|}{|A|}) \geq r. \tag{19}$$

We can prove, for any rough inclusion of the form of μ_T^I, the transitivity property cf. [8]:

$$\mu_T^I(u, v, r) \ \wedge \mu_T^I(v, w, s) \Rightarrow \mu_T^I(u, w, T(r, s)). \tag{20}$$

We recall the proof for the reader convenience.

Proof 4 *We begin with the observation that*

$$DIS(u, w) \subseteq DIS(u, v) \cup DIS(v, w) \tag{21}$$

hence

$$\frac{|DIS(u, w)|}{|A|} \leq \frac{|DIS(u, v)|}{|A|} + \frac{|DIS(v, w)|}{|A|} \tag{22}$$

We let

$$\begin{cases} g_T(\frac{|DIS(u,v)|}{|A|}) = r \\ g_T(\frac{|DIS(v,w)|}{|A|}) = s \\ g_T(\frac{|DIS(u,w)|}{|A|}) = t \end{cases} \tag{23}$$

Then

$$\begin{cases} \frac{|DIS(u,v)|}{|A|} = f_T(r) \\ \frac{|DIS(v,w)|}{|A|} = f_T(s) \\ \frac{|DIS(u,w)|}{|A|} = f_T(t) \end{cases} \quad (24)$$

Finally, by (22)

$$f_T(t) \leq f_T(r) + f_T(s) \quad (25)$$

hence

$$t = g_T(f_T(t)) \geq g_T(f_T(r) + f_T(s)) = T(r, s) \quad (26)$$

witnessing $\mu_T(u, w, T(r, s))$. *This concludes the proof.*

Let as observe also that each rough inclusion of the form of μ_T^I is symmetric:

$$\mu_T^I(x, y, r) \Leftrightarrow \mu_T^I(y, x, r). \quad (27)$$

We restrict ourselves to the above rough inclusions as model ones. Now, we consider a formal model of granulation of knowledge.

3 Granular computing: A model for granulation

Consider a universe U of things along with its sub-universe $I(U)$ of individual things endowed with a rough inclusion μ. For an individual thing $x \in I(U)$ and a real number $r \in [0, 1]$, we define the *granule of the radius r about x*, symbolically denoted $g(x, r, \mu)$, as follows:

$$g(x, r, \mu) = Cls(\Psi(x, r, \mu)), \quad (28)$$

where the collection $\Psi(x, r, \mu)$ is defined as:

$$y \in \Psi(x, r, \mu) \Leftrightarrow \mu(y, x, r). \quad (29)$$

Granules of knowledge defined formally as above play a crucial role in rough set approach to spatial reasoning and data mining cf. [8],[9]. It is therefore vital to be able to decide whether a given thing is an ingredient of a given granule. This problem constitutes the final section of this note.

4 The membership problem

Let us recall first that the rough inclusion μ_T^I on the universe of a given information system is T-transitive and symmetric for t-norm $T = L, P$. In this case we have the following proposition,

Proposition 4 *For each symmetric and f-transitive rough inclusion μ and each thing y in the universe of individual things on which μ is defined, $ingr(y, g(x, r, \mu))$ holds if and only if $\mu(y, x, r)$.*

We recall the proof for completeness of exposition.

Proof 5 *Recall that the granule $g(x, r, \mu)$ is defined as the class of the collection $\Psi(x, r, \mu)$ in 28. By property C2 of classes, $ingr(y, g(x, r, \mu))$ means that there is an individual thing z such that (i) $Ov(y, z)$ and (ii) $\mu(z, x, r)$. By (i) there exists an individual thing w such that (iii) $ingr(w, y)$ and (iv) $ingr(w, z)$ hence (v) $\mu(w, y, 1)$ and (vi) $\mu(w, z, 1)$. By symmetry of μ, (vii) $\mu(y, w, 1)$ hence by transitivity of μ, from (vi) and (vii) it follows (ix) $\mu(y, z, f(1, 1))$, i.e., $\mu(y, z, 1)$. From (ix) and (ii) one obtains by transitivity of μ that (x) $\mu(y, x, f(1, r))$, i.e., $\mu(y, x, r)$.*

By the last proposition, we are able to represent granules $g(x, r, \mu)$ for a transitive and symmetric rough inclusion μ as sets:

$$g(x, r, \mu) = \{y : \mu(y, x, r)\}. \tag{30}$$

5 A comment

To illustrate difficulties with the membership problem, let us consider the rough inclusion μ_M based on the Goedel residual implication. In this case, granules are given as:

$$g(x, r, \mu) = U \ \text{ in case } x \geq r \text{ and } Cls(\{z : z \leq x\}) \text{ in case } x < r, \tag{31}$$

and for the truth of the formula $ingr(y, g(x, r, \mu))$ in the latter case there should exist w, t such that $ingr(w, y)$, $ingr(w, t)$, $t \leq x$, i.e., $w \leq y$, $w \leq t$, $t \leq x$, i.e., any w with $w \leq y$, $w \leq x$ would do which implies that each y is an ingredient of the granule. The class propety C2 causes in this case the synergy effect involving each individual thing on the universe. The way out of this phenomenon is to adopt a new definition of a granule as a set of the form (30) for each rough inclusion μ.

6 Conclusion

We surveyed the problem of establishing ingredients of granules defined within rough mereological calculi. This problem is shown to be solved in case of transitive and symmetric rough inclusions while in more general cases ingredients of granules may exhaust the whole universe and have no relation to granules themselves. The problem solves positively in the most important case of Łukasiewicz rough inclusions μ_L^I in information/decision systems and in the case of the Łukasiewicz residual rough inclusion μ_L as they are symmetric and L-transitive. This singles out the Łukasiewicz t-norm as the one applicable in problems of approximate reasoning when granulation operations are applied cf. [9].

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